Mathematics PhD Qualifying Exam
Information Sheet and Instructions

Objective

Ensure that students have core mathematical skills that will facilitate solution of graduate-level engineering problems.

Instructions

- You have **120** minutes to complete the exam
- Turn in **ONLY SIX** (6) problems from **SIX** (6) different areas
- Submit each problem on a **SEPARATE** sheet of paper. Put your name on each page.
- **CLOSED** BOOK
- **CLOSED** NOTES
- **CALCULATORS** SUPPLIED BY THE DEPARTMENT are allowed (no function, graphing or matrix capability)
- Most of the grade will be based upon showing **STEP-BY-STEP WORKING** of the problems
- Members of the exam committee will be available to answer any **QUERIES**
- **Laplace Transform** Tables are appended to the exam and should be used for the problems in the Laplace Transform section; some other common formulas and identities are also supplied in the appendix.
- Cell phones and other electronic devices are not permitted in the exam room.
- Only the problems to be graded should be turned in.

Exam Topics and Learning Objectives

- BYU Undergraduate courses that cover many of these topics: MATH 112, 113, 302, 303, 313, 314, 334, 447, 510, 511 and ME EN 273.

1. **Differential Calculus:**
    - Representative course at BYU: Math 112.
    - Students will demonstrate the ability to graphically interpret functions
    - Students will be able to take partial derivatives and total differentials (including related techniques such as the chain rule)
    - Students will be able to derive and interpret Taylor series of functions

2. **Integral Calculus:**
    - Representative course at BYU: Math 113.
    - Students will be adept at integrating definite and indefinite integrals, including the use of integration by parts and change of variables
    - Students will be able to assess double and triple integrals (e.g. for volume of rotation of functions), and apply transformation of coordinates

3. **Matrices:**
    - Representative courses at BYU: Math 302, Math 303, Math 334.
    - Students will demonstrate ability to manipulate and evaluate matrices using a variety of operations (e.g. sums, products, transpose, inverse, determinants, characteristic values).
    - Students will demonstrate the ability solve systems of linear equations using matrix operations.

4. **Ordinary Differential Equations:**
    - Representative courses at BYU: Math 302, Math 303, Math 334.
    - Students will demonstrate the ability to determine general and particular solutions of linear and nonlinear first-order differential equations and linear second order differential equations with constant coefficients.
    - Students will formulate and solve differential equations representing physical systems, e.g., falling or sliding bodies, rate-limited systems, diffusion, or vibrating systems.

5. **Vector Calculus:**

- Representative courses at BYU: Math 302, Math 314
- Students will be proficient at vector algebra and calculus, including the ability to use operations involving gradient, curl, and divergence
- Students will be able to use Stokes' theorem and to perform line and surface integrals
- Students will be able to deal with mathematical operations on conservative, irrational, and solenoidal fields, and on scalar potentials

6. **Fourier Series:**
    - Representative course at BYU: Math 303.
    - Students will be able to generate Fourier or exponential series, including the coefficients, for arbitrary functions, and determine appropriate sine and cosine series of odd and even functions.
    - Students will demonstrate methods to test for convergence and rates of convergence for a given Fourier series.

7. **Laplace Transform:**
    - Representative courses at BYU: Math 303, Math 334.
    - Students will use Laplace transforms to solve ordinary differential equations with constant coefficients.
    - Students will use partial fraction expansion to assist in taking inverse Laplace transforms.
8. **Numerical Methods:**
    - Representative courses at BYU: ME EN 373, Math 510, Math 511.
    - Students will demonstrate an ability to interpolate, differentiate, integrate, functionally approximate, and find zeros of numerical data.
    - Students will demonstrate an ability to solve initial value, boundary value, and Eigen value problems as well as partial differential equations.

9. **Partial Differential Equations:**
    - Representative courses at BYU: Math 303, Math 447.
    - Students will demonstrate the ability to solve initial and/or boundary value problems for the Laplace, heat, and wave equations in two variables by the method of separation of variables.

SAMPLE EXAMS

Sample Exam 1 **(TRY THIS EXAM EARLY IN YOUR STUDIES)*:***

1: Differential calculus

1. A 20ft long ladder leans against a vertical building. If the bottom of the ladder slides away from the building horizonatally at a rate of 2 ft/sec, how fast is the ladder sliding on the building when the top of the ladder is 12 ft above the ground?

2: Integral calculus

1. A cylindrical tank 6 ft in diameter and 10 ft long is lying on its side. If the tank is half full of oil, weighing 58 lb/ft3, find the force, F, exerted by the oil on one of the circular end-faces of the tank.

3: Vector calculus

1. A particle moves in space so that a time, t, its position is given as $x=2t+3$, $y=t^2+3t$, $z=t^3+2t^2$. Find the components of its velocity and acceleration in the direction of the vector 2**i**+3**j**+4**k** when t=1.

4: Matrices and determinants

1. Solve the following system of equations using either Gauss-Jordan elimination or matrix inversion techniques. Show your work.

$$3x_1 + 4x_2 + 2x_3 = 5$$

$$x_1 - 2x_2 + 3x_3 = 12$$

$$-x_1 - x_2 - 2x_3 = -6$$

5: Numerical methods

1. Find the equation of the straight line that best fits the data given in the following table.

X	Y
-1.0	0.0
0.0	1.0
4.0	2.0
1.0	3.0
1.0	2.0
5.0	1.0

6: Fourier series

1. Find the general formula for the nth coefficients (both sine and cosine) of a Fourier series that represents the offset square wave function having a period of 2π.

$$y(x) = \begin{cases} 2 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

7: Laplace transforms

1. Solve $\frac{d^2\,y}{dt} + 4\frac{dy}{dt} = 8\sin 2t, y(0) = 0, \dot{y}(0) = 2$ using the attached table of Laplace transforms.

8: ODE's

1. Given the equation

$$\frac{1}{a}f(t) - \frac{k}{a}x = \ddot{x} + \frac{c}{a}\dot{x}$$

Determine the homogeneous solution
Assume $f(t) = \sum_{n=-\infty}^{\infty} Cn\,(\cos nwt + i\sin nwt)$. Find the particular solution.
What is the total solution?

9: PDE's

1. Given $u_t = au_{xx}$ find $u(x,t)$ given the following boundary coordinates:

$$u(0,t) = 1 \;\; u(l,t) = 1 \quad u(x,0) = x$$

Mechanical Engineering PhD Math Qualifier Equation Sheet

Double Angle Formulae

$$\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi$$
$$\cos(\theta \pm \varphi) = \cos\theta\cos\varphi \mp \sin\theta\sin\varphi$$
$$\tan(\theta \pm \varphi) = \frac{\tan\theta \pm \tan\varphi}{1 \mp \tan\theta\tan\varphi} \quad \left[\theta \pm \varphi \neq k + \tfrac{1}{2}\pi\right]$$
$$\sin(\theta) + \sin(\varphi) = 2\sin\tfrac{1}{2}(\theta+\varphi)\cos\tfrac{1}{2}(\theta-\varphi)$$
$$\sin(\theta) - \sin(\varphi) = 2\cos\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi)$$
$$\cos(\theta) + \cos(\varphi) = 2\cos\tfrac{1}{2}(\theta+\varphi)\cos\tfrac{1}{2}(\theta-\varphi)$$
$$\cos(\theta) - \cos(\varphi) = 2\sin\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi)$$

Cosine Formula (for triangle with internal angles, A,B,C, and opposite side lengths a,b,c):

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Certain Derivatives	
$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\ln(x)$	$1/x$
a^x	$\ln(a)\, a^x$

Certain Integrals	
$f(x)$	$\int f(x)dx$
$\tan(x)$	$\ln\lvert\sec(x)\rvert$
$\sin^2(x)$	$\tfrac{1}{2}(x - \tfrac{1}{2}\sin(2x))$
$\cos^2(x)$	$\tfrac{1}{2}(x + \tfrac{1}{2}\sin(2x))$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$

Laplace Transforms

	$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1.	1	$\frac{1}{s}$
2.	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\frac{1}{s-a}$
4.	$\sin kt$	$\frac{k}{s^2+k^2}$
5.	$\cos kt$	$\frac{s}{s^2+k^2}$
6.	$\sinh kt$	$\frac{k}{s^2-k^2}$
7.	$\cosh kt$	$\frac{s}{s^2-k^2}$
8.	$\delta(t-a)$	e^{-as}
Operational Properties		
9.	$e^{at} f(t)$	$F(s-a)$
10.	$f(t-a)\mathscr{U}(t-a), a > 0$	$e^{-as}F(s)$
11.	$t^n f(t), \quad n = 1, 2, 3, \ldots$	$(-1)^n \frac{d^n}{ds^n} F(s)$
12.	$f^{(n)}(t), \quad n = 1, 2, 3, \ldots$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
13.	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
14.	$\int_0^t f(\tau)g(1-\tau)d\tau$	$F(s)G(s)$
Some Consequences of the Above Functions		
15.	$t^n e^{at}, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
16.	$e^{at} \sin kt$	$\frac{k}{(s-a)^2+k^2}$
17.	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
18.	$t \sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
19.	$t \cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
20.	$\sin kt - kt \cos kt$	$\frac{2k^3}{(s^2+k^2)^2}$

21.	$\sin kt + kt\cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$

Sample Exam 2 **(TRY THIS EXAM WHEN YOU THINK YOU ARE READY)*:***

1: Differential calculus

1. Find $\frac{d^2y}{dx^2}$ $if\ x = t + 3t^3\ and\ y = t - 2t^2$.

2: Integral calculus

1. Find the volume under the paraboloid: $x^2+y^2=az$, above the xy-plane and inside the cylinder: $x^2+y^2=4ax$.

3: Vector calculus

1. Compute the divergence of

   $\mathbf{F}=x^2y\mathbf{i}+z\mathbf{j}+wyz\mathbf{k}$

   Recall that $div\ \mathbf{F} = \nabla \bullet \mathbf{F}$

4: Matrices and determinants

1. Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$$

5: Numerical methods

1. Evaluate the following integral using both 2- and 3- point Gauss quadrature, correct to four significant figures. Compare your answers to the exact answer obtained using calculus.

$$I = \int_0^{\pi/2} \sin\theta d\theta$$

To help you, the table on the next page gives Gauss points and weights.

TABLE 22.1 Weighting factors c and function arguments x used in Gauss-Legendre formulas.

Points	Weighting Factors	Function Arguments	Truncation Error
2	$c_0 = 1.0000000$ $c_1 = 1.0000000$	$x_0 = -0.577350269$ $x_1 = 0.577350269$	$\cong f^{(4)}(\xi)$
3	$c_0 = 0.5555556$ $c_1 = 0.8888889$ $c_2 = 0.5555556$	$x_0 = -0.774596669$ $x_1 = 0.0$ $x_2 = 0.774596669$	$\cong f^{(6)}(\xi)$
4	$c_0 = 0.3478548$ $c_1 = 0.6521452$ $c_2 = 0.6521452$ $c_3 = 0.3478548$	$x_0 = -0.861136312$ $x_1 = -0.339981044$ $x_2 = 0.339981044$ $x_3 = 0.861136312$	$\cong f^{(8)}(\xi)$
5	$c_0 = 0.2369269$ $c_1 = 0.4786287$ $c_2 = 0.5688889$ $c_3 = 0.4786287$ $c_4 = 0.2369269$	$x_0 = -0.906179846$ $x_1 = -0.538469310$ $x_2 = 0.0$ $x_3 = 0.538469310$ $x_4 = 0.906179846$	$\cong f^{(10)}(\xi)$
6	$c_0 = 0.1713245$ $c_1 = 0.3607616$ $c_2 = 0.4679139$ $c_3 = 0.4679139$ $c_4 = 0.3607616$ $c_5 = 0.1713245$	$x_0 = -0.932469514$ $x_1 = -0.661209386$ $x_2 = -0.238619186$ $x_3 = 0.238619186$ $x_4 = 0.661209386$ $x_5 = 0.932469514$	$\cong f^{(12)}(\xi)$

6: Fourier series

1. Find the Fourier series representation of $f(x) = |\sin(\pi x)|$.

7: Laplace transforms

1. Using Laplace transforms determine the response of the overdamped system modeled by the ordinary differential equation

$$\ddot{y} + 3\dot{y} + 2y = r(t); \quad \dot{y}(0) = 0, y(0) = 0$$

where

$$r(t) = \begin{cases} 1, & \text{if } 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

8: ODE's

1. The equation of motion for a parachutist is given by

$$m\dot{v} = mg - bv$$

where *v* is velocity, *m* is the mass of the parachutist, *b* is a linear drag coefficient for the parachute, and *g* is the acceleration due to gravity. If the parachute opens at time $t=t_0$, and the velocity at time t_0 is v_0, find an equation giving v for all times greater than t_0.

9: PDE's

1. Given $\nabla^2 u = u_{tt}$ where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ solve for u given a one dimensional spatial case and boundary condition:

$u(0, t)= A$ $\quad$ $u(l, t)=B$

$u(x,0)=x$ $\quad$ $u_t(x,0)=x^2$

Mechanical Engineering PhD Math Qualifier Equation Sheet

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Certain Integrals	
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$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$

$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|$

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11.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n\frac{d^n}{ds^n}F(s)$
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13.	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
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Some Consequences of the Above Functions		
15.	$t^ne^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$
16.	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
17.	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
18.	$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
19.	$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$

20.	$\sin kt - kt\cos kt$	$\dfrac{2k^3}{(s^2+k^2)^2}$
21.	$\sin kt + kt\cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$

Sample Exam 3 (ADDITION QUESTIONS FOR EACH TOPIC):

1: Differential calculus

1. Given: $y^3 + x^2y^4 + x^3 = 1$, find $\frac{dy}{dx}$

2. Prove that the curves
   $5y - 2x + y^3 - x^2y = 0$
   and
   $2y + 5x + x^4 - x^3y^2 - 0$
   intersect at right angles at the origin.
   Note: if any two curves intersect at right angles at a given point, then the product of the slopes equals -1.

3. A ball is placed on the following surface at point (2,4,8). What direction will the ball roll?
   $$x^3 - 4x^2y + 2y^2x - z = 0$$

4. A light 4 miles from a straight shoreline makes 5 revolutions per minute. How fast is the light moving along the shore when the beam makes an angle of $\pi/4$ with the shoreline?

5. Using the ideal gas law (i.e., *PV=kT* with *k=10*), find the rate at which the temperature is changing at the instant when the volume of the gas is 100 cubic inches and the pressure of the gas is 10 pounds per square inch. Assume the volume of the gas is increasing at the rate of 3 cubic inches per second and the pressure is decreasing at the rate of 0.2 pounds per square inch per second.

6. Find by implicit differentiation: $D\frac{d^2y}{dx^2}y = \frac{d^2y}{dx^2}$ when $7x^3 + 5y^2 = 23$.

7. When helium expands adiabatically, its pressure is related to its volume by the formula
   $P*V^{1.67}$ =constant.
   At a certain time, the volume of the helium in a balloon is 18 m^3 and the pressure is 0.3 kg/m^2. If the pressure is increasing at a rate of 0.01 kg/m^2/sec, how fast is the volume changing? Is the volume increasing or decreasing?
8. Find by implicit differentiation: $\frac{d^2y}{dx^2}$ $when\ 7x^3 + 5y^2 = 23$.

2: Integral calculus

1. Evaluate the expression:

$$F(t) = \int_0^t x\sqrt{a^2 - x^2}\, dx \quad \text{where } |t| \leq a$$

2. Integrate:

$$\int e^{ax} \sin(3x)\, dx$$

3. If a particle travels in a straight line with a speed $|v| = t^3 - 3t^2 + 5\sin(t)$ in meters/second, what is the distance between its positions at t=2 and t=5 seconds?

4. Calculate the volume within the cylinder $x^2 + y^2 = b^2$ between the planes z=0 and $y + z = a^2$ given that $a^2 \geq b \geq 0$.

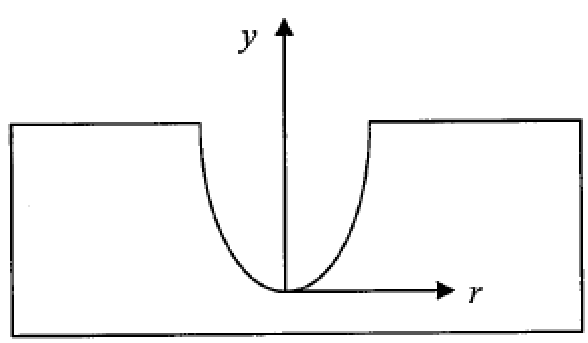
5. Find the mass of a right triangular plate with legs **a** and **b**, if the density is proportional to the square of the distance from the vertex of the right angle.

6. Evaluate the integral $\int x^2 e^x\, dx$.

7. You have been asked to design a mold for a new chocolate candy. The candy will be in the shape of a parabola of revolution. The equation of the parabola is

$$y=1.5r^2$$

where *y* is the height above the bottom of the mold, and *r* is the radius of the surface of the mold, in mm.



If the candy is to have a total volume of 5 cm^3, how deep should the mold be?

8. Find the limits of integration for

$$\iiint_V f(x, y, z)\, dV$$

when the region V is bounded by the paraboloid $x^2+4y^2=16-z$ and the xy- plane, and lies in the first octant. What is the volume *accurate to one decimal place*?

3: Vector calculus

1. Find the directional derivative of f at P in the direction of **a**, where
$$f = x^2 + y^2, \qquad P: (1,1), \qquad \mathbf{a} = \mathbf{3i} - \mathbf{4j}$$
2. Find the curl v of, where
$$v = x^2 \sin(y)\,\mathrm{i} + 3\mathrm{zxj} - zy^2 k$$
3. The acceleration of a particle at any time $t \geq 0$ is given by
$$\bar{a} = \frac{d\bar{v}}{dt} = (12\cos(2t))\hat{i} - (8\sin(2t-1))\hat{j} + (16t)\hat{k}$$
If the initial velocity vector is [0,8,0] and the initial displacement vector is [10,0,0], find an expression for the velocity and displacement for all time.
4. Find the total work done in moving a particle in a force field given by
$$F = 5xyz\hat{i} - 3z\hat{j} + 8xz\hat{k}$$
Along the curve $x = t^2 + 1, y = 3t, z = t$ from $1 \leq t \leq 2$.
5. The acceleration of a particle at any time $t \geq 0$ is given by
$$\bar{a} = \frac{d\bar{v}}{dt} = (12\cos(2t))\hat{i} - (8\sin(2t-1))\hat{j} + (16t)\hat{k}$$
If the initial velocity vector is [0,8,0] and the initial displacement vector is [10,0,0], find an expression for the velocity and displacement for all time.

6. Compute the divergence of
$$\mathbf{F} = x^2 y\mathbf{i} + z\mathbf{j} + xyz\mathbf{k}$$
Recall that *div* $\mathbf{F} = \nabla \bullet \mathbf{F}$.

7. The equation of a plane in space can be described by the equation
*ax+by+cz+d=*0
or by the vector $\mathbf{n} \bullet \mathbf{x} = \mathrm{D}$. In the vector equation **n** is a unit normal to the plane, **x** is any point in the plane, and D is the minimum distance of the plane from the origin of the *x-y-z* reference axes. Given a plane described by
5x-6y+10z-20=0,
Determine **n, x,** and D, and then determine how far point (10, -12, 15) is from this plane (the perpendicular distance of this point form the plane).

4: Matrices and determinants

1. Find the solution to the following matrix problem (show all your work):

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -5 & 7 \\ 4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

2. Find the determinant of the following matrix (show all your work):

$$\begin{bmatrix} 3 & 5 & 2 \\ 4 & 2 & 3 \\ -1 & 2 & 4 \end{bmatrix}$$

   Is the matrix invertible?

3. Find the equation of a plane that goes through (0,0,1), (1,1,1) and (2,3,0), using Cramer's Theorem for solving linear equations by determinants. The general equation of a plane is

$$ax + by + cz = d$$

   Show your work for full credit.

4. Find the eigenvalues and the eigenvectors for the following matrix:

$$\begin{bmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Determine a) the rank of the matrix A, and b) the determinant of A where

$$A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ 1 & 4 & 3 & 0 \\ 1 & 5 & 5 & 4 \\ 1 & 6 & -2 & -3 \end{bmatrix}.$$

6. Determine the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

5: Numerical methods

1. Using the data below, estimate

$$\int_0^2 y\mathrm{d}x$$

using the trapezoidal rule.

2. For the differential equation

$$7\dot{x} + t\exp(x) = 0$$

where *x* is the dependent variable, and *t* is the independent variable, estimate the value of *x*(0.5) using two iterations of Euler's method (step size of 0.25). *x*(0)=3.

3. Solve

$$\frac{dy}{dx} = x + y \qquad y(o) = 1, h = 0.1$$

using a fourth-order Runga-Kutta method. Iterate until the numerical solution matches the analytic solution to the fifth decimal place in the interval from 0.0 to 0.3. The analytical solution is

$$y = 2e^x - x - 1$$

$e^{0.0}$=1.0000
$e^{0.1}$=1.10517
$e^{0.2}$=1.22140
$e^{0.3}$=1.34986

4. Using Simpson's 1/3 rule, numerically approximate the function below (with constant, s) and give an estimate of the error. Use a step size of 0.25. Show your work.

$$\int_2^6 \frac{s}{(1 - x^4)} dx$$

x	y
0.0	1.000
0.5	1.414
1.0	2.000
1.5	2.828
2.0	4.000

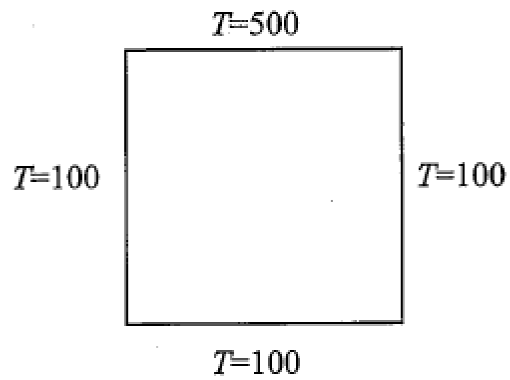
5. Using Newton-Raphson iteration, find one root of the equation

$$f(x) = x^3 - x^2 - 10x - 8.$$

Perform two iterations with three significant digits of precision. Use *x*=6 as an initial estimate of the root.

6. A two-dimensional square block of width *x*=*y*=1 has its sides held at the temperatures given below. Assuming steady state, no heat generation and constant properties, the governing equation is given by

$$\frac{\partial^2 T}{\partial^2 x}+\frac{\partial^2 T}{\partial^2 y}=0.$$



Show how you would obtain the temperature distribution in the block by modeling it numerically using finite difference equations. Note: You do not have to solve for the distribution! Rather show how to a) derive the equation for an interior node and b) handle the boundary conditions.

6: Fourier series

1. Find the Fourier sine series representation of $f(x) = x, 0 < x < 1$.
2. Let the function *f(x)* be defined by:

   $f(x) = 3x^8 - x^6 - 17x^2 + \cos^2 x$ -$(-\pi \le x \le \pi)$

   on the given interval. Outside this interval, the function is periodic such that $f(x) = f(x + 2\pi)$. What is the value of the coefficient of the <u>first sine term</u> in the Fourier series representing this function?
3. For the function

   $$f(x) = x$$

   The Fourier Series is

   $$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\,x}{(2n-1)^2}$$

   Differentiate this series term by term to obtain a new series. Give the formula for the new series and explain what function it represents.
4. Obtain the Fourier Sine series for $f(x) = \cos \pi x$.
   Note: $((2sinAcosB) = \sin(A+B) + \sin(A-B))$
5. Find the Fourier series for the periodic square-wave function

   $$f(t) = \begin{cases} 0 \; when -2 < t < -1 \\ k \; when -1 < t < 1 \\ 0 \; when \; 1 < t < 2 \end{cases}$$

   Use a basic sine series, cosine series and Euler's formula to derive the complex form of the Fourier series.

7: Laplace transforms

1. Solve the initial value problem using Laplace Transforms:
   $y''(t) - 6y'(t) + 9y(t) = t^2e^{3t}$ $\quad\quad y(0) = 2, y'(0) = 6$
2. Solve the initial value problem using Laplace Transforms:
   $\frac{d^2y}{dt^2} + \frac{dy}{dt} = \begin{cases} 2\sin t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$ $\quad\quad y(0) = 0, \dot{y}(0)=0$
3. Solve the following initial value problem using the attached table of Laplace transforms:
   $\ddot{y} - 3\dot{y} + 2y = 4t + e^{3t};$ $\quad\quad y(0) = 1, \quad \dot{y}(0) = -1$

8: ODE's

1. Find the general solution to the equation: $\frac{d^4y}{dx^4} - 3\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$
2. An object of mass *m*=(0.5g) kg (where g is 9.81) stretches a vertical spring by $L = \frac{8}{9}\mathrm{m}$ under gravity. There is no damping, and gravity is the only force. The mass is displaced 0.5m upwards from its equilibrium position and given an initial velocity of 1m/s downwards. Find the displacement, *u(t)*, at any subsequent time *t*.

   Note that the equation of motion is given by: $mu'' + cu' + ku = 0$, where *m* is the mass, *c* is the damping coefficient and *k* is the spring coefficient given by *k=mg/L*.

3. Given the following equation

   $$Ax^2u'' + Bxu' + Cu = f(x)$$

   Find only the homogenous solution using the assumption that $u(x) = x^\lambda$ where A=-3, B=5, C=-9, and $f(x) = xe^{u'}$
4. Solve the following ordinary differential equation:
   $$y''' + 2y'' - y' - 2y = 2 - 4x^3$$

   $$y(0) = -10, \quad y'(0) = -6, \quad y''(0) = 12.$$

   Note that the characteristic roots (eigenvalues) for this equation are all integers.
5. Solve the differential equation
   $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$$
   on the domain [0,*L*] given that *y* = 0 when *x* = 0 and *y* = 0 when *x* = *L*.
6. Using matrix methods, solve the system of ODEs given by $\dot{y} = Ay$ where $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

What are the eigenvalues and eigenvectors of A?

7. Solve the nonhomogeneous equation

$$y'' - y' - 2y = 10\cos x, \qquad y(0) = 1, \qquad y'(0) = 0.$$

9: PDE's

1. Solve the equation $\frac{\partial^2 u}{\partial x\,\partial y} = \sin x \cos y$ subject to the boundary conditions that at $y = \frac{\pi}{2}, u = 2x \text{ and at } x = \pi, u = 2\sin y.$
2. A string is stretched between the fixed points 0 and 1 on the x-axis and released at rest from the position $y = B\sin 2\pi x$ where B is constant. Find an expression for the subsequent displacement y(x,t).

3. Heat is generated uniformly at a constant rate per unit volume throughout a semi-infinite slab $0 \le x \le x \le \pi$ that is initially at temperature f(x) and whose faces x=0 and x=π are kept at temperature zero. Set up and solve for the temperature distribution in the slab throughout time.
4. The transverse vibrations of a string are given to be $\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial^2 t}$
   i. Solve for y by finding the specific solution that satisfies the following conditions

$$y = \frac{2b}{l}x \qquad 0 < x < \frac{1}{2}$$

$$y = \frac{2b}{l}x + 2b \qquad \frac{l}{2} < x < l$$

And $\frac{\partial y}{\partial t} = 0$ at t=0.

Mechanical Engineering PhD Math Qualifier Equation Sheet

Double Angle Formulae

$$\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi$$

$$\cos(\theta \pm \varphi) = \cos\theta\cos\varphi \mp \sin\theta\sin\varphi$$

$$\tan(\theta \pm \varphi) = \frac{\tan\theta \pm \tan\varphi}{1 \mp \tan\theta\tan\varphi} \quad \left[\theta \pm \varphi \neq k + \tfrac{1}{2}\pi\right]$$

$$\sin(\theta) + \sin(\varphi) = 2\sin\tfrac{1}{2}(\theta + \varphi)\cos\tfrac{1}{2}(\theta - \varphi)$$

$$\sin(\theta) - \sin(\varphi) = 2\cos\tfrac{1}{2}(\theta + \varphi)\sin\tfrac{1}{2}(\theta - \varphi)$$

$$\cos(\theta) + \cos(\varphi) = 2\cos\tfrac{1}{2}(\theta + \varphi)\cos\tfrac{1}{2}(\theta - \varphi)$$

$$\cos(\theta) - \cos(\varphi) = 2\sin\tfrac{1}{2}(\theta + \varphi)\sin\tfrac{1}{2}(\theta - \varphi)$$

Cosine Formula (for triangle with internal angles, A,B,C, and opposite side lengths a,b,c):

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Certain Derivatives	
$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\ln(x)$	$1/x$
a^x	$\ln(a)\, a^x$

Certain Integrals	
$f(x)$	$\int f(x)dx$
$\tan(x)$	$\ln\lvert\sec(x)\rvert$
$\sin^2(x)$	$\frac{1}{2}(x - \frac{1}{2}\sin(2x))$
$\cos^2(x)$	$\frac{1}{2}(x + \frac{1}{2}\sin(2x))$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x - a}{x + a}\right\rvert$

Laplace Transforms		
	$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1.	1	$\frac{1}{s}$
2.	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\frac{1}{s-a}$
4.	sin kt	$\frac{k}{s^2+k^2}$
5.	cos kt	$\frac{s}{s^2+k^2}$
6.	sinh kt	$\frac{k}{s^2-k^2}$
7.	cosh kt	$\frac{s}{s^2-k^2}$
8.	$d(t-a)$	e^{-as}
Operational Properties		
9.	$e^{at} f(t)$	$F(s-a)$
10.	$f(t-a)\mathscr{U}(t-a), a > 0$	$e^{-as}F(s)$
11.	$t^n f(t), \quad n = 1, 2, 3, \ldots$	$(-1)^n \frac{d^n}{ds^n} F(s)$
12.	$f^{(n)}(t), \quad n = 1, 2, 3, \ldots$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
13.	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
14.	$\int_0^t f(\tau)g(1-\tau)d\tau$	$F(s)G(s)$
Some Consequences of the Above Functions		
15.	$t^n e^{at}, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
16.	$e^{at} \sin kt$	$\frac{k}{(s-a)^2+k^2}$
17.	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
18.	$t \sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
19.	$t \cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
20.	$\sin kt - kt \cos kt$	$\frac{2k^3}{(s^2+k^2)^2}$

21.	$\sin kt + kt\cos kt$	$\frac{2ks^2}{(s^2+k^2)^2}$